

Viscous accretion discs around rotating black holes

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ABSTRACT

The stationary hydrodynamic equations for transonic viscous accretion discs in Kerr geometry are derived. The consistent formulation is given for the viscous angular momentum transport and the boundary conditions on the horizon of a central black hole. An expression for the thickness of the disc is obtained from the vertical Euler equation for general accretion flows with vanishing vertical velocity. Different solution topologies are identified, characterized by a sonic transition close to or far from the marginally stable orbit. A numerical method is presented that allows to integrate the structure equations of transonic accretion flows. Global polytropic solutions for the disc structure are calculated, covering each topology and a wide range of physical conditions. These solutions generally possess a sub-Keplerian angular momentum distribution and have maximum temperatures in the range $10^{11} - 10^{12}$ K. Accretion discs around rotating black holes are hotter and deposit less angular momentum on the central object than accretion discs around Schwarzschild black holes.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – relativity – methods: numerical – X-rays:general

1 INTRODUCTION

The presence of hot optically thin gas in accreting black hole systems such as X-ray binaries or active galactic nuclei has been established by X-ray and γ -ray observations. The spectra are very hard, with power-law photon indices $\alpha_N \sim 1.5 - 2.5$ in the energy range $E \sim \text{keV}$ to 100 keV (Gilfanov et al. 1995; Maisack et al. 1993; Johnson et al. 1994). They are usually fitted by a Sunyaev-Titarchuk (1980) thermal comptonization model with optical depth $\tau_{\text{es}} \gtrsim 1$ and electron temperature $T_e \sim 10^9$ K. These hard spectral components exclude as an interpretation all optically thick accretion models, like the standard thin accretion disc model of Novikov & Thorne (1973; hereafter NT73) or the 'slim accretion disc' model of Abramowicz et al. (1988), both of which cannot attain temperatures higher than a few $\sim 10^7$ K. The first hot optically thin disc model in agreement with hard spectral components was the two-temperature accretion disc model by Shapiro, Lightman & Eardley (1976), with electron temperature $T_e \sim 10^8 - 10^9$ K. However, this model was found to be thermally unstable due to the temperature-density dependence of the cooling rate by local bremsstrahlung emission. A related branch of hot optically thin disc solutions was found, when radiative cooling channels were extended towards higher temperature domains, including synchrotron radiation and comptonization of synchrotron and bremsstrahlung photons. These solutions are thermally and viscously stable for long-wavelength perturbations. The first models of this class were local thermal equilibrium models, i.e. the equation of energy balance is solved locally for any radius from a given underlying disc structure. Narayan & Yi (1994, 1995a, 1995b) use a self-similar disc structure with sub-Keplerian angular momentum distribution (Begelman & Meier 1982; Spruit et al. 1987), whereas Abramowicz et al. (1995) and Chen (1995) assume a slim disc structure with Keplerian angular momentum distribution. In spite of these kinematic and other minor differences in the particular cooling functions, both approaches yield the same local thermal equilibria (Chen et al. 1995). Advection becomes relevant if temperatures are very high, such that the relative half-thickness of the disc, h/r , is not negligibly small. This is the case in hot optically thin disc flows, where possible heating mechanisms primarily heat the ions to nearly virial temperature. Since Coulomb collisions become inefficient the

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electrons gain only little of the thermal energy of the ions and remain at temperatures of $T_e \sim 10^9 - 10^{10}$ K due to inefficient cooling. The local thermal equilibrium solutions are in general globally advection-dominated. Morphologically they resemble the ion-supported torus model of Rees et al. (1982). It turns out (Abramowicz et al. 1995) that the branch of stable optically thin equilibrium solutions is limited to a maximum accretion rate $\dot{M}_{\max} \sim \alpha^2$, as long as the viscosity parameter α is smaller than some critical value α_{crit} , depending on the details of the model (Chen et al. 1995).

Recently, global solutions of hot optically thin accretion discs have been calculated. This requires to solve the full system of coupled equations of motion for the fluid in the disc, causing considerably higher numerical efforts due to the sonic point in the radial Euler equation. Viscous transonic solutions were calculated within the pseudo-Newtonian potential (Paczynski & Wiita 1980) for wedge-shaped flows by Chakrabarti (1996) and for disc-like flows by Chen et al. (1996) and Narayan et al. (1996). It turned out, that for most radii, the local thermal equilibrium solutions yield a good approximation for the temperature distribution in the disc. At the inner edge of the accretion disc, located at the horizon of the black hole, matter is forced to corotate with the black hole surface. This condition can not be implemented properly within the pseudo-Newtonian potential formalism. Therefore, disc properties in the very vicinity of the black hole are to be studied in full relativity. Furthermore both, stellar mass black hole candidates in galactic binary systems as well as the putative supermassive black holes in the centers of active galactic nuclei are expected to generally possess a considerable amount of angular momentum. In the first case the hole is spun up by accretion through the disc, whereas in the latter case the existence of jets gives theoretical support for a rapidly rotating central object (e.g. Blandford 1977; Rees et al. 1982).

Accretion flows around rotating black holes have to be studied in Kerr geometry. A set of equations for stationary disc-like viscous transonic accretion flows around rotating black holes has been proposed by Lasota (1994; hereafter L94). Global transonic solutions of this system, corrected for some misprints, were given by Peitz (1994) for a polytropic equation of state. Solutions of a related set of equations, describing hot optically thin discs cooled by bremsstrahlung emission, have recently been calculated by Abramowicz et al. (1996, hereafter ACGL96) and discussed by Lasota & Abramowicz (1996). So far, most formulations for angular momentum transport have been using approximations for the shear, which are inconsistent with the kinematics in the supersonic inner part of the accretion discs.

In this paper we derive the system of equations governing stationary viscous transonic disc flows in Kerr geometry (sections 2, 3). In particular, we give the consistent expression for the equation of angular momentum balance, important for the boundary condition at the horizon. We then discuss in section 4 the properties of critical points in inviscid transonic disc-like flows, which hold as well for viscous transonic flows in the limit of small viscosity. In section 5 we describe a numerical method to solve the structure equations for transonic flows and present global solutions for a polytropic equation of state.

2 BASIC EQUATIONS

2.1 Metric

The half-thickness h of the disc is assumed to satisfy the condition $h/r \lesssim 1$. Then a power series expansion of the Kerr metric written in cylindrical coordinates $\{t, \phi, r, z\}$, to zeroth order in the small quantity z/r , yields a first approximation for the underlying disc metric. In geometrical units ($G = c = 1$), which are used throughout this paper, this metric \mathbf{g} reads (NT73)

$$\mathbf{g} = -\frac{r^2 \Delta}{A} dt^2 + \frac{A}{r^2} (d\phi - \omega dt)^2 + \frac{r^2}{\Delta} dr^2 + dz^2, \quad (1)$$

where

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2, \\ A &= r^4 + r^2 a^2 + 2Mr a^2, \\ \omega &= -g_{t\phi}/g_{\phi\phi} = 2Mar/A. \end{aligned}$$

Here M is the black hole mass and $a = J/M$ its specific angular momentum. Δ vanishes at the event horizon, given by $r_h = M + (M^2 - a^2)^{1/2}$. A subscript 'h' refers to quantities evaluated at r_h .

2.2 Conservation laws

Primary variables are the particle current \mathbf{n} , the stress-energy tensor \mathbf{T} and the entropy flux \mathbf{s} . The motion of the fluid is governed by conservation of \mathbf{n} and \mathbf{T} , written as

$$0 = \nabla \cdot \mathbf{n}, \quad (2)$$

$$0 = \mathbf{h} \cdot (\nabla \cdot \mathbf{T}), \quad (3)$$

$$0 = \mathbf{u} \cdot (\nabla \cdot \mathbf{T}), \quad (4)$$

where \mathbf{u} is the four velocity of the fluid and $\mathbf{h} = \mathbf{u} \otimes \mathbf{u} + \mathbf{g}$ projects into its *local rest frame* (LRF). Equations (2), (3), (4) are the continuity equation, the Navier-Stokes equations and the energy equation, respectively.

The particle current is $\mathbf{n} = n\mathbf{u}$ with the baryon number density n . The stress-energy tensor for a single component fluid with viscosity and heat flux is

$$\mathbf{T} = (\rho + p)\mathbf{u} \otimes \mathbf{u} + p\mathbf{g} + \mathbf{t} + \mathbf{q} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{q} . \quad (5)$$

Here $\rho = \rho_0 + \varepsilon$ is the total energy density, $\rho_0 = nm_B$ is the baryon rest mass density (m_B being the mean baryon rest mass), ε is the internal energy density and p the isotropic pressure. All thermodynamic quantities refer to the LRF. Viscous contributions to \mathbf{T} are contained in the viscous stress tensor \mathbf{t} , given by

$$\mathbf{t} = -\zeta\Theta\mathbf{h} - 2\eta\boldsymbol{\sigma} , \quad (6)$$

where $\boldsymbol{\sigma}$ is the shear tensor, the scalar functions η and ζ are the shear and bulk viscosity coefficients, respectively, and $\Theta = \nabla \cdot \mathbf{u}$ is the expansion of the fluid world lines. The energy flux relative to the LRF is described by the spacelike vector \mathbf{q} . The four-entropy flux is defined as

$$\mathbf{s} = ns\mathbf{u} + \mathbf{q}/T , \quad (7)$$

with s the specific entropy density and T the thermodynamic temperature.

Stationarity and axial symmetry of \mathbf{g} imply the existence of two commuting Killing vector fields, $\mathbf{k} = \partial_t$ and $\mathbf{m} = \partial_\phi$, which give rise to two conserved currents,

$$\mathbf{P} = \mathbf{T} \cdot \mathbf{k} , \quad \nabla \cdot \mathbf{P} = 0 , \quad (8)$$

$$\mathbf{J} = \mathbf{T} \cdot \mathbf{m} , \quad \nabla \cdot \mathbf{J} = 0 . \quad (9)$$

For ideal flows ($\mathbf{t} = \mathbf{q} = 0$), Eqs. (8), (9) yield two constants of motion, namely energy $E = -\mu u_t$ and angular momentum $L = \mu u_\phi$, where u_t is the binding energy and $\mu = (\rho + p)/\rho_0$ the specific relativistic enthalpy. The specific angular momentum $\lambda = L/E = -u_\phi/u_t$ is introduced for later use.

2.3 Velocity field and reference frames

We consider accretion flows with a four velocity field $\mathbf{u} = (u^t, \Omega u^t, u^r, 0)$, where $\Omega = u^\phi/u^t = d\phi/dt$ is the angular velocity and $u^r = dr/d\tau$ the radial velocity component, negative for accretion. In addition to the LRF attached to the fluid in motion with \mathbf{u} , two other reference frames are frequently used in relativistic accretion disc theory.

The *locally non-rotating frame* (LNRF) is characterized by $u_\phi = 0$ and orbiting with angular velocity ω . Tensor indices with hat ' $\hat{}$ ' refer to LNRF. The physical 3-velocity with respect to LNRF has azimuthal and radial components

$$v^{(\hat{\phi})} = \frac{A\tilde{\Omega}}{r^2\Delta^{1/2}} , \quad v^{(\hat{r})} = \frac{u^r}{\gamma\mathcal{U}} , \quad (10)$$

where $\tilde{\Omega} \equiv \Omega - \omega$ and $\mathcal{U}^2 \equiv h^{rr} = (u^r)^2 + \Delta/r^2$. Here

$$\gamma = \left(1 - (v^{(\hat{\phi})})^2\right)^{-1/2} = \left(1 - \frac{A^2\tilde{\Omega}^2}{r^4\Delta}\right)^{-1/2} \quad (11)$$

is the azimuthal Lorentz factor w.r.t. LNRF. Using γ allows to write the normalization $\mathbf{u} \cdot \mathbf{u} = -1$ as

$$u^t = \frac{\gamma A^{1/2}\mathcal{U}}{\Delta} , \quad u_\phi = \frac{\gamma A^{3/2}\mathcal{U}\tilde{\Omega}}{r^2\Delta} . \quad (12)$$

In the following, we frequently write $u \equiv u^r$ and $l \equiv u_\phi$. In terms of these variables, γ and $\tilde{\Omega}$ are

$$\gamma = \left(1 + \frac{\Delta l^2}{A\mathcal{U}^2}\right)^{1/2} , \quad \tilde{\Omega} = \frac{r^2\Delta l}{A(A\mathcal{U}^2 + \Delta l^2)^{1/2}} . \quad (13)$$

The *corotating frame* (CRF) is orbiting with the angular velocity Ω of the fluid, being related to the LNRF by a Lorentz boost with γ in the azimuthal direction. Tensor indices in CRF are labelled by bar ' $\bar{}$ '. The radial component of the physical 3-velocity w.r.t. CRF is $v^{(\bar{r})} = \gamma v^{(\hat{r})} = u/\mathcal{U}$.

The *dragging of inertial frames* forces any matter at the horizon to corotate with the surface of the black hole, $\tilde{\Omega}_h = 0$ or $\gamma_h = 1$, which follows from Eq. (13). Consequently, $v^{(\bar{r})}_h$ and $v^{(\hat{r})}_h$ are both equal to the speed of light by virtue of the metric. This is in contrast to accretion onto non-collapsed objects, where corotation with its solid surface is imposed as a boundary condition. For later reference we further introduce the total Lorentz factor w.r.t. LNRF,

$$\gamma_t = \left(1 - (v^{(\hat{\phi})})^2 - (v^{(\hat{r})})^2\right)^{-1/2} = \gamma(1 - (v^{(\bar{r})})^2)^{-1/2} , \quad (14)$$

which is singular at the horizon due to $v^{(\bar{r})}_h = 1$.

3 EQUATIONS OF DISC STRUCTURE

The flow is treated as a 1-D stationary problem, where the disc midplane coincides with the equatorial plane of the hole, mass increase of the hole and self-gravity of the disc are neglected. We use vertically integrated thermodynamic variables. Then the equations governing radial and vertical disc structure decouple.

3.1 Continuity equation

From the continuity equation $(nu^\alpha)_{;\alpha} = 0$ it follows that the total flux of particles through a surface of constant r is a constant, the accretion rate \dot{M} , given by

$$\dot{M} = -2\pi r u \Sigma_0, \quad (15)$$

where

$$\Sigma_0 = \int_{-h}^h \rho_0 dz \simeq 2h\rho_0 \quad (16)$$

is the surface rest mass density. For later use we introduce two other vertically integrated thermodynamic quantities, namely the total surface density, $\Sigma \simeq 2h\rho$, and the vertically integrated pressure, $P \simeq 2hp$.

3.2 Radial momentum equation

The radial Navier-Stokes equation $h_\alpha^r T_{;\beta}^{\alpha\beta} = 0$ can be written as

$$u'u + u^2\mathcal{G} + \mathcal{U}^2\mathcal{H} = -(\rho + p)^{-1}(\mathcal{U}^2 p' + S_r + Q_r), \quad (17)$$

with \mathcal{G} , \mathcal{H} , \mathcal{K} defined by

$$\mathcal{G} = (\ln \sqrt{g_{rr}})_{,r} = (a^2 - Mr)/(r\Delta), \quad (18)$$

$$\mathcal{H} = \gamma^2 AM\mathcal{K}/(r^2\Delta)^2, \quad (19)$$

$$\mathcal{K} = (\Omega - \Omega_k^+)(\Omega - \Omega_k^-)/(\Omega_k^+ \Omega_k^-). \quad (20)$$

The angular velocities

$$\Omega_k^\pm = \pm \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}} \quad (21)$$

correspond to prograde (+) and retrograde (−) Keplerian orbits. Partial derivatives w.r.t. r are denoted by a prime. The quantity \mathcal{K} measures the deviation of the angular velocity from the Keplerian value, with limiting cases $\mathcal{K}(\Omega = \Omega_k^\pm) = 0$ and $\mathcal{K}(\Omega = 0) = 1$. The quantities $S_r = h_{r\alpha} t_{;\beta}^{\alpha\beta}$ and $Q_r = h_{r\alpha} (q^\alpha u^\beta + q^\beta u^\alpha)_{;\beta}$ describe contributions due to viscosity and heat flux. In the following we assume these forces to be small compared to pressure forces. Then the radial Navier-Stokes equation (17) reduces to the radial Euler equation for a perfect fluid. With the help of the continuity equation, and after integration over the disc height, the radial Euler equation can be written as

$$\frac{u'}{u} = \frac{\mathcal{N}}{\mathcal{D}}, \quad \mathcal{N} = \frac{c_{\text{ad}}^2}{r} - \frac{u^2}{\mathcal{U}^2} \mathcal{G} - \mathcal{H}, \quad \mathcal{D} = \frac{u^2}{\mathcal{U}^2} - c_{\text{ad}}^2, \quad (22)$$

where c_{ad} is the adiabatic sound speed. Eq. (22) possesses a critical point for $\mathcal{D} = 0$, identified with a sonic point in the CRF. Since causality requires $c_{\text{ad}} < 1$, the conditions $|v^{(\bar{r})}_{\text{h}}| = 1$ and $|v^{(\bar{r})}| < c_{\text{ad}}$ at r_{out} suffice to conclude, that black hole accretion is transonic (Shapiro & Teukolsky 1983). Thus any global solution must satisfy

$$\mathcal{N}(\mathcal{D} = 0) = 0 \quad (23)$$

as a regularity condition.

3.3 Conservation of angular momentum

The azimuthal Navier-Stokes equation can be expressed in the form of angular momentum conservation $J_{;\alpha}^\alpha = 0$, where the conserved current J^r is given by

$$J^r = (\rho + p)u^r u_\phi + t_\phi^r + q^r u_\phi + q_\phi u^r. \quad (24)$$

Neglecting the angular momentum associated with the heat flux and using the continuity equation yields

$$0 = \rho_0 r u^r (\mu u_\phi)_{,r} + (r t_\phi^r)_{,r}. \quad (25)$$

The vertically integrated angular momentum balance follows after integration over r ,

$$\frac{\dot{M}}{2\pi}(L - L_0) = 2hrt_\phi^r, \quad (26)$$

where L_0 is an integration constant. We did not neglect angular momentum associated with the internal energy. Therefore Eq. (26) is the general equation of angular momentum balance (apart from a contribution due to photons) for any temperature, not restricted to flows with $\mu \simeq 1$ as in NT73 or L94.

3.4 Description of shear tensor and viscosity

We assume the coefficient of bulk viscosity to vanish ($\zeta = 0$). The viscous stress tensor then reduces to $\mathbf{t} = -2\eta\boldsymbol{\sigma}$. The shear tensor component σ_ϕ^r can be written as (Eq. (A4))

$$2\sigma_\phi^r = \mathcal{U}^2 l \left(\frac{l'}{l} + \mathcal{H} \right) + \frac{u^2 l}{3} \left(\frac{\mathcal{N}}{\mathcal{D}} - \frac{2}{r} + 3\mathcal{G} \right) + 2 \left[(Ma^2 - r^3)\Omega - Ma \right] \frac{\gamma A^{1/2} \mathcal{U}}{r^4}. \quad (27)$$

As demonstrated in App. A, expression (27) may be approximated by

$$2\sigma_\phi^r = \mathcal{U}^2 l' + 2 \left[(Ma^2 - r^3)\Omega - Ma \right] \gamma A^{1/2} \mathcal{U} r^{-4}. \quad (28)$$

Based on the dimensional estimate for the viscosity due to the motion of turbulent fluid elements in the disc, $\eta \sim \rho_0 v_{\text{turb}} l_{\text{turb}}$, we use a parametrization for the kinematic viscosity coefficient $\nu = \eta/\rho_0$, given by

$$\nu = \alpha c_{\text{ad}} r^\beta f_c, \quad (29)$$

where α is the viscosity parameter and $l_{\text{turb}} \sim r^\beta$ allows to mimic the standard assumption $l_{\text{turb}} \sim h$ as well as other dependencies, such as $l_{\text{turb}} \sim P/P'$ (Papaloizou & Stanley 1986). Viscosity is a transport process and operates on a finite time-scale, τ_ν , limited by causality, no matter whether it is mediated by turbulent gas motion, magnetic fields, or photons. τ_ν becomes equal or larger than the accretion time-scale τ_{acc} as the accreted material approaches the horizon, and as a consequence the viscosity vanishes there,

$$\nu_h = 0. \quad (30)$$

A cut-off function f_c has been introduced to guarantee such a behaviour. We choose (Narayan 1992)

$$f_c = \begin{cases} \left(1 - (v^{(\bar{r})}/c_\nu)^2 \right)^2 & ; \quad |v^{(\bar{r})}| \leq c_\nu, \\ 0 & ; \quad |v^{(\bar{r})}| > c_\nu, \end{cases} \quad (31)$$

parametrized by the maximum velocity of viscous information transport, c_ν . Since $\nu_h = 0$, viscous stresses t_ϕ^r vanish at the horizon (*no-torque* condition; Abramowicz & Prasanna 1990) and the integration constant L_0 in Eq. (26) is the angular momentum accreted by the black hole. Its value is obtained as part of the solution.

In the limiting case of a Keplerian disc, i.e. for $r \geq r_{\text{ms}}$ and $u^r \rightarrow 0$, $u_{,r}^r \rightarrow 0$, $l \rightarrow l_k$, $\Omega \rightarrow \Omega_k$, Eqs. (27), (28) both reduce to the expression of NT73,

$$2\sigma_\phi^r = \gamma_k^3 A^{3/2} \Delta^{1/2} r^{-5} \Omega'_k, \quad (32)$$

where $\gamma_k = \gamma(\Omega_k)$ denotes the Keplerian Lorentz factor w.r.t. LNRF. For non-Keplerian accretion flows, L94 proposed to replace γ_k in Eq. (32) by the azimuthal Lorentz factor w.r.t. LNRF, γ , as in Eq. (11) and Ω'_k by Ω' . Since $\gamma_h = 1$, the resulting expression vanishes at the horizon, and thus leads to a *no-torque* condition as well. It does so by constraining the kinematics instead of using the physically motivated condition (30). In addition, this expression leads to a reversal of shear in the vicinity of the black hole, where Ω' changes sign. Recently, ACGL96 replaced γ in the L94 expression by the total Lorentz factor w.r.t. LNRF, γ_t , as in Eq. (14). Different expressions for σ_ϕ^r are compared in App. A.

3.5 Conservation of energy

The energy conservation equation $u^\alpha T_{\alpha;\beta}^\beta = 0$ can be combined with the continuity equation to

$$\rho_0 u^r T s_{,r} = -\sigma_{\alpha\beta} t^{\alpha\beta} - 1/3 \Theta t_\alpha^\alpha - q^\alpha a_\alpha - q_{;\alpha}^\alpha, \quad (33)$$

where $\mathbf{a} = \nabla_{\mathbf{u}} \mathbf{u}$ is the four-acceleration of the fluid. The compressive heating rate $-1/3 \Theta t_\alpha^\alpha$ vanishes due to the assumption $\zeta = 0$. The special relativistic term $\mathbf{a} \cdot \mathbf{q}$ describes the inertia of the heat flux. Neglecting this term, the vertically integrated energy equation is given by

$$Q_{\text{adv}} = Q_+ - Q_-, \quad (34)$$

where $Q_{\text{adv}} = u\Sigma_0 T s_{,r}$ describes the advection of entropy by the radial motion of the fluid, $Q_+ = -2h\sigma_{\alpha\beta}t^{\alpha\beta}$ is the rate of viscous heat generation and $Q_- = -2hq^z - 2hr^{-1}(rq^r)_{,r}$ is the local cooling rate.

3.6 Equation of state

As a first approximation we assume a polytropic relation between the vertically integrated pressure P and the surface rest mass density Σ_0 (Popham & Narayan 1991),

$$P = K\Sigma_0^\Gamma, \quad (35)$$

where $K(s)$ is a constant measuring the entropy of the flow and the polytropic index for a two-dimensional flow, $N = 1/(\Gamma - 1)$, is constant. The adiabatic sound speed may then be written as $c_{\text{ad}}^2 = (dP/d\Sigma)_{\text{ad}}$, leading to

$$c_{\text{ad}}^2 = \left(\frac{dP}{d\Sigma_0} \right)_{\text{ad}} \frac{\Sigma_0}{\Sigma + P} = \frac{\Gamma P / \Sigma_0}{1 + N\Gamma P / \Sigma_0}.$$

Using the continuity equation then yields

$$c_{\text{ad}}^2 = (N + \dot{\mathcal{M}}^{-1/N} (r|u|)^{1/N})^{-1}. \quad (36)$$

The 'modified accretion rate'

$$\dot{\mathcal{M}} = \Gamma^N K^N (\dot{M}/2\pi) \quad (37)$$

consists of a combination of constants \dot{M} , K and Γ and thus is a constant. For fixed \dot{M} and Γ , $\dot{\mathcal{M}}$ measures the entropy in the flow. For fixed K and Γ , $\dot{\mathcal{M}}$ is a measure for the accretion rate. For $\Gamma > 1$, the specific relativistic enthalpy in terms of c_{ad} is $\mu = (1 - Nc_{\text{ad}}^2)^{-1}$. The polytropic equation of state directly couples the temperature to the kinematics of the flow, and makes the energy equation obsolete. Instead of two independent parameters \dot{M} and K , one is left with a single parameter $\dot{\mathcal{M}}$.

3.7 Vertical disc structure

The fluid is assumed to be in vertical hydrostatic equilibrium, i.e. in the LRF the vertical pressure gradient $p_{,z}$ balances the vertical tidal acceleration towards the midplane of the disc. A proper treatment of this force balance requires an expansion of the z -component of the Navier-Stokes equation to $\mathcal{O}(z/r)^1$. As shown by Riffert & Herold (1995), in a Keplerian disc the hydrostatic equilibrium, obtained from the vertical Euler equation in this way, differs considerably from the expression given by NT73. For general disc flows with $\mathbf{u} = (u^t, u^\phi, u^r, 0)$ the vertical Euler equation $^{\ddagger} h_\alpha^z T_{;\beta}^{\alpha\beta} = 0$, expanded to $\mathcal{O}(z/r)^1$, may be written as (App. B)

$$\frac{1}{\rho + p} \frac{\partial p}{\partial z} = -\frac{z}{r} \left[\frac{M}{r^2} \tilde{\mathcal{R}} + c_{\text{ad}}^2 \mathcal{Z} \left(\frac{\mathcal{N}}{\mathcal{D}} + \frac{1}{r} \right) \right], \quad (38)$$

$$\tilde{\mathcal{R}} = [\mathcal{Y}(1 - a\Omega)^2 - 4a\Omega(1 - a\Omega) + 2r^2\Omega^2] \frac{A\gamma^2 \mathcal{U}^2}{\Delta^2} - \frac{r^2}{\Delta} u^2, \quad (39)$$

$$\mathcal{Y} = 1 - \frac{2M}{r} + \frac{3a^2}{r^2}, \quad \mathcal{Z} = \frac{2M}{r} - \frac{a^2}{r^2}. \quad (40)$$

It can be shown that $\tilde{\mathcal{R}}$ remains finite at r_h and therefore (38) is regular everywhere. In the limiting case of a (cold) Keplerian disc flow, e.g. for $c_{\text{ad}} \rightarrow 0$, $u \rightarrow 0$, $\Omega \rightarrow \Omega_k$, $\rho + p \rightarrow \rho_0$, Eq. (38) reduces to the formula given by Riffert & Herold (1995). To obtain an explicit expression for the half-thickness h from Eq. (38), we substitute $\partial p / \partial z \simeq p/h$ and assume

$$\frac{p}{\rho + p} \simeq \frac{\Sigma_0}{\Sigma + P} \frac{P}{\Sigma_0} \simeq \frac{\Sigma_0}{\Sigma + P} \left(\frac{dP}{d\Sigma_0} \right)_{\text{ad}} = c_{\text{ad}}^2. \quad (41)$$

This leads to

$$\frac{h^2}{r^2} = \frac{c_{\text{ad}}^2 r}{M\tilde{\mathcal{R}} + c_{\text{ad}}^2 r^2 \mathcal{Z}(\mathcal{N}/\mathcal{D} + 1/r)}. \quad (42)$$

If the radial pressure gradient in the vertical Euler equation is negligible, (42) reduces to $h^2/r^2 = (r/M)c_{\text{ad}}^2 \tilde{\mathcal{R}}^{-1}$.

The incomplete NT73 expression for the vertical hydrostatic equilibrium in Keplerian flows, given for comparison with alternative expressions for the disc height, reads

$$\frac{1}{\rho_0} \frac{\partial p}{\partial z} = -\frac{z}{r} \frac{M}{r^2} \mathcal{R}, \quad \mathcal{R} = \gamma_k^2 \left[\frac{(r^2 + a^2)^2 + 2\Delta a^2}{(r^2 + a^2)^2 - \Delta a^2} \right], \quad (43)$$

[‡] The derivation of the thickness of transonic accretion discs from the vertical Euler equation was proposed by J.P. Lasota, who also drew our attention towards a different approach by Abramowicz et al. (1996).

where γ_k is the Keplerian Lorentz factor w.r.t. LNRF. For non-Keplerian accretion discs, L94 proposed to replace γ_k in Eq. (43) by the azimuthal Lorentz factor (11). Recently, ACGL96 used the total Lorentz factor (14) instead of γ_k . Since γ_t is singular at the horizon, the disc half-thickness h , as derived with this approximation, vanishes at r_h , which leads to infinite densities and therefore to difficulties in the energy equation. The relative disk height (42), as well as alternative expressions, are shown in Fig. 12. Equation (42) yields the complete expression as long as the vertical velocity component u^z can be neglected.

4 SOLUTION TOPOLOGIES

The regularity condition at the sonic point allows to identify different solution topologies for accretion flows (Abramowicz & Zurek1981). This analysis, performed for a pseudo-Newtonian potential, as well as related studies for wedge-shaped flows in Kerr geometry (Chakrabarti1990) are generalized in this section. A subscript 's' refers to 'critical quantities', i.e. quantities at r_s .

As shown in App. C, the regularity condition $\mathcal{N}(\mathcal{D} = 0) = 0$ selects a 2-D hypersurface \mathcal{F} of the parameter space spanned by three critical variables, e.g. by (r_s, c_s, Ω_s) or by (r_s, E_s, λ_s) . Projections of \mathcal{F} onto the E_s - λ_s plane are plotted in Fig. 1 for a non-rotating and for a rotating ($a = 0.95$) black hole. For E_s, λ_s within ABC , three solutions for r_s exist. These are referred to as *inner* critical point, $r_{s,\text{in}}$, *intermediate* critical point, $r_{s,\text{int}}$, and *outer* critical point, $r_{s,\text{out}}$, according to their relative location. For E_s, λ_s beyond the lines BC and BA , only one solution for r_s exists, corresponding to an *inner* critical point for E_s, λ_s within I and to an *outer* critical point for E_s, λ_s within O . *Inner* critical points are located in the vicinity of the marginally stable orbit r_{ms} ($r_{\text{ms}} = 6r_g$ for $a = 0$ and $r_{\text{ms}} \simeq 1.94r_g$ for $a = 0.95$, $r_g = M$ being the gravitational radius), whereas *outer* critical points are located farther away from the black hole (App. C). The type of critical points depends on the critical slope $(d\lambda/dr)_s$, and separates unphysical from possible physical solutions.

In ideal flows, with $\lambda = \lambda_s$ and $E = E_s$ being conserved, $r_{s,\text{int}}$ is an center-type critical point. Any other critical points are saddle-type, corresponding to possible physical solutions. Ideal solutions are determined by two out of three constants of motion E, λ, \dot{M} . For given E and λ , the corresponding accretion rate \dot{M} increases as r_s decreases. For any λ within ABC , there exists an energy E_m such that the corresponding \dot{M} for a solution passing $r_{s,\text{in}}$ is equal to \dot{M} for a solution passing $r_{s,\text{out}}$, $\dot{M}_m(E_m, \lambda, r_{s,\text{in}}) = \dot{M}_m(E_m, \lambda, r_{s,\text{out}})$ (Chakrabarti1990). Thus the corresponding line BD indicates, where stationary solutions actually 'jump' between different topologies as E, λ are varied continuously. In the following, solutions with $E_s > E_m$ are referred to as type I, and solutions with $E_s < E_m$ as type II. Solutions with $E_s = E_m$ pass through both $r_{s,\text{in}}$ and $r_{s,\text{out}}$ and are therefore global accretion solutions only if a shock occurs within $r_{s,\text{in}}$ and $r_{s,\text{out}}$. The dashed line in Fig. 1 indicates the maximum λ which a global ideal transonic flow with a given E may have in order to pass the centrifugal barrier.

Viscous solutions with non-conserved E and λ are represented by curves $E(\lambda)$. The location of r_s results from the position of E_s and λ_s on that curve, which is generally unknown in advance. The solution topology at r_s , depending on $(d\lambda/dr)_s$, is now determined by the equation of angular momentum balance, introducing additional parameters, e.g. α and L_0 . For moderate $(d\lambda/dr)_s$, the topological classification in terms of local critical quantities E_s and λ_s yields essentially the same result as for ideal flows. The $E(\lambda)$ curves of such flows may then be superposed on Fig. 1. If $E(\lambda)$ never crosses the line BD , the solution topology and the approximate location of r_s are known. This situation applies e.g. for quasi-ideal flows with globally small gradients dE/dr and $d\lambda/dr$, where $E(\lambda)$ is bound in narrow regions around E_s and λ_s . If $E(\lambda)$ crosses the line BD , the unknown position of E_s and λ_s becomes relevant. In special cases E_s and λ_s are restricted, e.g. if the flow is assumed ideal within r_s .

Figure 1 shows that flows around rotating black holes possess generally higher energies and thus higher temperatures than flows around non-rotating black holes with corresponding topology.

5 GLOBAL SOLUTIONS

5.1 Model equations and numerical method

The system of two coupled ODEs solved in this section consists of Eq. (22) and Eq. (26) combined with Eq. (28),

$$\frac{du}{dr} = \frac{u\mathcal{N}}{\mathcal{D}}, \quad (44)$$

$$\frac{dl}{dr} = \frac{u}{U^2} \frac{(L - L_0)}{\nu} + 2 \left[Ma + (r^3 - Ma^2)\Omega \right] \frac{\gamma A^{1/2}}{Ur^4}. \quad (45)$$

The adiabatic sound speed c_{ad} is calculated according to Eq. (36), assuming $N = 3/2$ ($\Gamma = 5/3$). The kinematic viscosity coefficient ν is parametrized by Eq. (29), using $c_\nu = 1$ and $\beta = 1$. The inner edge of the disc is located at $r_{\text{in}} = r_h$, the outer edge at $r_{\text{out}} = 100r_g$. The radial Euler equation has a sonic point at radius r_s , which is calculated as part of the solution by the condition $\mathcal{D} = 0$. This provides, together with the regularity condition $\mathcal{N}(\mathcal{D} = 0) = 0$, the boundary condition for Eq.

(44). At r_{in} the condition $v_h = 0$ requires $L_h = L_0$, which is used as a boundary condition. The four boundary conditions for the four unknowns u , l , r_s , L_0 are therefore given by $\mathcal{D} = 0$, $\mathcal{N} = 0$, $L_h = L_0$ and e.g. $\lambda(r_{\text{out}}) = \lambda_{\text{out}}$.

Eqs. (44), (45) are mapped onto the unit interval in x , separately for the sub- and supersonic domain. The resulting four equations are then solved with x as an independent variable. The sonic point is mapped onto $x = 0$, where $\mathcal{N} = 0$, $\mathcal{D} = 0$ are taken as boundary conditions both for the supersonic and subsonic equations. The horizon and the outer radius of the disc are both mapped onto $x = 1$, where $L = L_0$ is used as boundary condition for the supersonic equations. For a given L_0 , the solution is then uniquely determined, including λ_{out} . An iteration scheme in L_0 is set up to find global solutions for a given λ_{out} . The boundary value problem is integrated by a collocation method (Bader & Kunkel 1989), using a fully adaptive grid with variable number of subintervals. This allows to obtain the solution to a pre-specified accuracy. A parameter continuation is used to study the variation in a , α and \dot{M} , as well as for the iteration in L_0 .

We use dimensionless quantities by normalizing distances to the gravitational radius r_g and velocities to the speed of light (geometrical units). For a polytropic equation of state the resulting system is independent of mass M . We present ideal and viscous solutions for any of the topologies discussed in Sect. 4. The relative half-thickness h/r of the disc is calculated from Eq. (42). No retrograde disc flows with $\lambda < 0$ are considered.

5.2 Ideal solutions

To give examples for all solution topologies, we plot model sequences of increasing λ for $E = 1.015$ in the case of a non-rotating ($a = 0$) black hole (see Fig. 2) and for $E = 1.06$ in the case of a rotating ($a = 0.95$) black hole (see Fig. 3). The various models correspond to crosses \times in Fig. 1. For lower λ , within ABD or O , solutions pass an *outer* critical point. For λ within O , u grows with decreasing r . For λ within ABD , the existence of the *inner* and *inter* sonic point becomes 'visible' in a deformation of u at the corresponding locations. As λ approaches the line BD from within ABD , u passes the *outer* sonic point followed by a local maximum and a local minimum, both within the supersonic region. This behavior occurs because $\mathcal{N} = 0$ is encountered in total three times, first together with $\mathcal{D} = 0$ at the sonic point, and thereafter two more times for finite \mathcal{D} . As λ crosses the line BD , stationary solutions 'jump' from type II to type I topology, passing the *inner* critical point (compare e.g. Chakrabarti 1990; Kafatos & Yang 1994). As λ approaches the line BD from within BCD , u first passes a local maximum followed by a local minimum, both within the subsonic region, before passing the *inner* sonic point. The majority of type I solutions with λ within I have again monotonic u -profiles.

The 'jump' of stationary solutions at the line BD is not visible in the corresponding Ω -profiles. In the vicinity of the horizon, the Ω -distribution shows a kinematic boundary layer structure, dropping with a steep gradient to $\tilde{\Omega}_h = 0$. In that kinematic boundary layer, the maximum of Ω and the steepness of $d\Omega/dr$ depend on the angular momentum of the flow and the rotation of the hole. This is illustrated in Fig. 4, where Ω is plotted for a sequence of type I models with increasing a and fixed $E = 1.2$, $\lambda = 1.0$.

5.3 Viscous solutions

For given boundary conditions the global solution is uniquely determined and possesses either type I or type II topology. Type I solutions are generally found for moderate α and higher \dot{M} . In Fig. 5 we plot type I solutions around a non-rotating black hole, which differ only in α , with otherwise fixed a , \dot{M} and L_0 . They show a globally sub-Keplerian angular momentum distribution. With increasing α the angular momentum transport becomes more efficient, such that practically all reasonable boundary conditions λ_{out} , from $\lambda_{\text{out}} \gtrsim \lambda_s$ to $\lambda_{\text{out}} \lesssim \lambda_k(r_{\text{out}})$ are met. For α fixed, λ_{out} increases with L_0 . This is demonstrated in Fig. 6 for a sequence of type I solutions around a rotating ($a = 0.95$) black hole, which are for the same a , α and \dot{M} , but differ in L_0 . There exist solutions which approach a Keplerian distribution λ_k globally or even exceed λ_k locally. These require high λ in the vicinity of r_{ms} , obtained for lower α and possess higher L_0 . In Fig. 7 we plot two examples of such trans-Keplerian solutions for $a = 0$. Solutions of that kind possess extremely high temperature at r_{out} , such that their physical relevance appears doubtful. For very low viscosity, e.g. for $\alpha = 10^{-4} - 10^{-3}$, accretion profiles are quasi-ideal. Under these conditions, type I solutions with $E(\lambda)$ within BCD can be calculated, possibly passing local extrema in their u -profiles. In particular, the quasi-constant λ may exceed $\lambda_k(r_{\text{ms}})$ in the vicinity of r_{ms} . In Fig. 8 we plot examples of type I quasi-ideal solutions for $a = 0$, with globally sub-Keplerian as well as locally super-Keplerian λ -distribution. For higher λ or for energies below unity, $r_{s,\text{out}}$ is shifted outwards, in extreme cases beyond r_{out} (Fig. 14). Then even solutions with inverted angular momentum transport in their sub-Keplerian outer parts exist (Fig. 8), as reported by Abramowicz et al. (1988). However, in order to guarantee that these are global solutions, the integration must be extended to larger r_{out} .

Except from quantitative differences, type I solutions show the same qualitative behavior for any a . The c_{ad} -distribution and thus the temperature distribution possess a maximum outside r_s . For given a , \dot{M} and L_0 , higher α yield steeper λ -profiles and higher temperatures. Therefore, quasi-ideal flows represent the limit of lowest temperature discs for a set of a , \dot{M} and L_0 . Since temperature remains high at r_{out} , no (cold) standard thin disc configuration is met. This reproduces a well-known property of hot accretion flows, and is not an artefact of a polytropic equation of state. A common feature of type I solutions

is a very flat λ -distribution inside r_s . This is due to low ν resulting from the decrease of c_{ad} within r_s . The energy in the disc increases moderately towards r_{out} . The decrease of c_{ad} towards r_h is imprinted on the E -profiles and the L -profiles (not plotted), which show steeper gradients than λ in the vicinity of r_h . Thus, in general, even for quasi-flat λ -distribution inside r_s , the angular momentum L_0 deposited onto the black hole in a type I flow is significantly lower than L_s . The relative disc height respects $h/r \lesssim 1$ at r_s , but in models with effective angular momentum transport exceeds unity beyond $\simeq 20r_g$ due to very high temperatures. The Ω -distributions show a kinematic boundary layer structure similar to ideal flows as a consequence of flat λ -distributions within r_s , such that disc models with $\nu = 0$ in the supersonic domain (Popham & Narayan 1992), essentially yield similar profiles.

Viscous solutions passing the *outer* sonic point exist for higher α and lower \dot{M} . Since the angular momentum at $r_{s,out}$ is strongly sub-Keplerian, type II solutions possess smaller L_0 . In Fig. 9 we plot a sequence of type II solutions around a non-rotating black hole, for fixed \dot{M} and with different L_0 . Since $r_s \gg r_h$, global solutions with any reasonable boundary condition λ_{out} do not exist for each α . The two solutions with higher λ in Fig. 9 are for the same $\alpha = 0.06$, chosen such that the corresponding maximum possible λ_{out} is Keplerian. Lower α only allow for $\lambda_{out} < \lambda_k(r_{out})$ (for otherwise fixed parameters). The two solutions with lower λ in Fig. 9 are for $\alpha = 0.05$. The global $\lambda = 0$ distribution with $L_0 = 0$ does no more depend on α , describing the limit of 'spherical disc accretion' around a non-rotating hole. This limit is reproduced for any E and λ within Fig. 1. Alternatively, as for type I solutions, different λ_{out} can be found for fixed a , \dot{M} and L_0 by variation of α . This requires \dot{M} high enough in order to achieve $r_s < r_{out}$. A sequence of such type II solutions around a rotating ($a = 0.95$) black hole is shown in Fig. 10. As for type I flows, quasi-ideal solutions with $E(\lambda)$ within ABD exist, again similar to corresponding ideal solutions and not shown here.

Solutions around rotating and non-rotating black holes differ only quantitatively. Since the λ -distribution is strongly sub-Keplerian at r_s , accretion flows with smaller λ_{out} result in general and no type II solution with a global λ_k -distribution does exist. Nevertheless, λ can be steeper within r_s than for type I solutions due to higher α . Type II solutions have generally lower temperature than type I, with a maximum at the horizon. However, as before, no (cold) standard disc configuration is matched at r_{out} . With decreasing \dot{M} , r_s moves outward and c_{ad} decreases due to lower λ_s and thus less efficient global angular momentum transport. Therefore the relative disc height h/r exceeds unity only in the very outer disc regions. The kinematic boundary layer is less prominent due to generally lower λ . We do not find solutions with negative accreted angular momentum.

6 SUMMARY AND DISCUSSION

We derived the equations of motion for stationary viscous disc-like accretion flows around rotating black holes. In particular, we give a complete expression for the shear tensor that governs the transport of angular momentum in relativistic transonic disc flows. In order not to violate causality, the kinematic viscosity has to vanish at the horizon. This is achieved by an appropriate parametrization. Consequently, no torque is exerted on the horizon. It is shown that the expression widely used for the shear, calculated by assuming circular trajectories, is incompatible with the kinematics in the supersonic inner part of the accretion discs. We wish to stress that it is important to use the correct boundary conditions at the horizon, even if it turns out that they do not strongly affect the global properties of solutions. The range of parameters, for which global solutions exist, can be affected by the choice of boundary conditions. Furthermore, we calculate an expression for the half-thickness of the disc, obtained by integration of the vertical Euler equation over disc height.

The properties of the sonic point are exploited to classify different solution topologies of ideal disc-like accretion flows around rotating black holes. Depending on angular momentum and energy, global solutions pass through the inner or outer critical point, referred to as type I and type II, respectively. We use a polytropic equation of state for vertically integrated pressure and density to calculate global viscous solutions. The coupled disc equations are solved between the horizon of the black hole and $100r_g$. A numerical method is presented that allows to integrate the structure equations of transonic accretion flows. In cases of moderate angular momentum transport, the topological classification for ideal flows also holds for viscous flows. We give examples of each topology for non-rotating and rapidly rotating black holes ($a = 0.95$). We present solutions, where the angular momentum at the outer edge of the disc varies between zero and the Keplerian value, for a wide range of parameters. The angular momentum L_0 accreted by the black hole is calculated as part of the solution. For small values of α and higher \dot{M} , solutions are of type I, with the sonic point located in the vicinity of the marginally stable orbit. In these cases, the temperature distribution in the disc has a maximum outside the sonic point. There exist solutions which are sub-Keplerian everywhere (Figs. 5, 6), and accretion flows, which are super-Keplerian in some region inside and outside the marginally stable orbit (Fig. 7, 8). In this case, the angular momentum accreted by the black hole generally exceeds the Keplerian value at the marginally stable orbit. For these partially super-Keplerian flows the sonic point is located between the photon orbit and the marginally stable orbit. For a given angular momentum at the outer edge of the disc, trans-Keplerian flows possess a higher temperature than sub-Keplerian flows. For higher values of α and smaller \dot{M} , solutions are of type II, with the sonic point located further away from the black hole. They generally possess a sub-Keplerian angular momentum

distribution for any radius. These solutions are colder than type I and have their maximum temperature at the horizon. In the outermost part of type I solutions, the relative disc height h/r exceeds unity, or the temperature reaches the virial temperature. Our solutions however remain consistent for the inner $20r_g$.

We obtain solutions around rotating black holes for any Kerr parameter a . Discs around rotating black holes are hotter and deposit less angular momentum onto the central object than discs around non-rotating black holes. For the latter the temperature maximum is broader and at larger radii than in the case of a rapidly rotating black hole. Temperatures of our solutions lie in the range of $10^{11} - 10^{12}$ K. They remain high throughout the whole disc and our solutions do not match a (cold) standard thin disc configuration at the outer edge. These results suggest to compare our polytropic model with other hot accretion disc models, e.g. with optically thin advective disc models. Global solutions of this type, described by a related system of equations, have been calculated recently (Abramowicz et al.1996). It turns out that the general features of both models agree. A polytropic equation of state therefore provides a reasonable description for many properties of hot optically thin advective disc flows. Since we do not find any solution with considerably lower temperature than $\sim 10^{11}$ K, a polytropic equation of state does not seem to reproduce the class of cold radiation-pressure supported optically thick slim discs (Abramowicz et al.1988).

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APPENDIX A: SHEAR TENSOR COMPONENT σ_ϕ^R

To calculate the shear tensor σ it is convenient to start from the decomposition of the gradient of the four-velocity, $\nabla \mathbf{u}$, into irreducible tensorial parts,

$$u_{\alpha;\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - a_\alpha u_\beta, \quad (\text{A1})$$

where ω is the antisymmetric rotation 2-form, σ is the tracefree symmetric shear tensor and \mathbf{a} is the four-acceleration of the fluid. The expansion of the fluid world lines is given by $\Theta = u^\alpha_{;\alpha} = u^r_{;r} + u^r/r$. Symmetrization (expressed by brackets) of Eq. (A1) leads to

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} + a_{(\alpha}u_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}. \quad (\text{A2})$$

The relevant component for the angular momentum equation is then

$$2\sigma_\phi^r = u^r_{;\phi} + g^{rr}u_{\phi;r} + a^r u_\phi + a_\phi u^r - \frac{2}{3}\Theta u^r u_\phi, \quad (\text{A3})$$

which, for $\mathbf{u} = (u^t, \Omega u^t, u^r, 0)$, can be written as

$$2\sigma_\phi^r = \mathcal{U}^2 l \left(\frac{l'}{l} + \mathcal{H} \right) + \frac{u^2 l}{3} \left(\frac{u'}{u} - \frac{2}{r} + 3\mathcal{G} \right) + 2 [(Ma^2 - r^3)\Omega - Ma] \frac{\gamma A^{1/2} \mathcal{U}}{r^4}. \quad (\text{A4})$$

Instead of using this full expression (with $u'/u = \mathcal{N}/\mathcal{D}$), σ_ϕ^r may be approximated by a simplification of the velocity field \mathbf{u} involved in the calculation of σ_ϕ^r . The simplest approximation of a Keplerian disc flow with $u^r = 0$ leads to Eq. (32). While consistent for nearly circular trajectories given in Keplerian discs (NT73), Eq. (32) breaks down in the inner part of transonic discs, where radial and azimuthal velocities become comparable.

A more appropriate simplification of Eq. (A4) consists of dropping terms which contain u'/u . We therefore neglect the expansion ($\Theta = 0$) and the radial four-acceleration ($a^r = 0$). This approximation yields

$$2\sigma_\phi^r = \mathcal{U}^2 l' + 2 [(Ma^2 - r^3)\Omega - Ma] \gamma A^{1/2} \mathcal{U} r^{-4}. \quad (\text{A5})$$

We found that viscous solutions calculated using Eq. (A5) are consistent with the assumptions made above for all parameters and for almost any radius. In Fig. 11 the exact expression (A4) was used to calculate a typical type II solution. This solution was used to calculate the approximation (A5), as well as other expressions proposed for the shear. The exact expression (A4) is plotted as a solid line and remains finite at the horizon. The approximation (A5) is plotted as a short dashed line and yields a good approximation to (A4) for all radii. Eq. (32) with γ and Ω' instead of γ_k and Ω'_k (long dashed line; L94) differs considerably, especially in the vicinity of the kinematic boundary layer. Eq. (32) with γ_t and Ω' instead of γ_k and Ω'_k (short dash-long dashed line; ACGL96) remains a good approximation for larger radii, but tends to extremely high values of inversed shear in the inner parts of the flow, due to the singular behaviour of γ_t at r_h . For larger radii all expressions approach the Newtonian limit.

APPENDIX B: VERTICAL EULER EQUATION

In this section the vertical Euler equation $h_\alpha^z T_{;\beta}^{\alpha\beta} = 0$ is derived for a velocity field $\mathbf{u} = (u^t, u^\phi, u^r, 0)$, to an accuracy of first order in (z/r) . This requires an expansion of the Kerr metric to $\mathcal{O}(z/r)^2$ (Riffert & Herold 1995), denoted by \mathbf{G} , and written as

$$\mathbf{G} = \mathbf{g} + \frac{z}{r} \tilde{\mathbf{g}} + \frac{z^2}{r^2} \tilde{\tilde{\mathbf{g}}}, \quad (\text{B1})$$

where \mathbf{g} is the zeroth order (Eq. (1); NT73), with non-vanishing components

$$\begin{aligned} g_{tt} &= \frac{A\omega^2}{r^2} - \frac{\Delta r^2}{A} = -\left(1 - \frac{2M}{r}\right), & g_{t\phi} = g_{\phi t} &= -\frac{A\omega}{r^2} = -\frac{2Ma}{r} \\ g_{\phi\phi} &= \frac{A}{r^2}, & g_{rr} &= \frac{\Delta}{r^2}, & g_{zz} &= 1, \end{aligned} \quad (\text{B2})$$

$\tilde{\mathbf{g}}$ is the first order correction, with non-vanishing components

$$\tilde{g}_{rz} = \tilde{g}_{zr} = g_{rr} \left(\frac{2M}{r} - \frac{a^2}{r^2} \right), \quad (\text{B3})$$

and $\tilde{\tilde{\mathbf{g}}}$ is the second order correction, with non-vanishing components

$$\begin{aligned} \tilde{\tilde{g}}_{tt} &= -\frac{M}{r} \left(1 + \frac{2a^2}{r^2} \right), \\ \tilde{\tilde{g}}_{t\phi} &= \tilde{\tilde{g}}_{\phi t} = -g_{t\phi} \left(\frac{3}{2} + \frac{a^2}{r^2} \right), \\ \tilde{\tilde{g}}_{\phi\phi} &= -a^2 \left(1 + \frac{5M}{r} + \frac{2Ma^2}{r^3} \right), \\ \tilde{\tilde{g}}_{rr} &= -g_{rr}^2 \left(\frac{3M}{r} - \frac{4M^2}{r^2} - \frac{a^2}{r^2} \left(3 - \frac{6M}{r} + \frac{2a^2}{r^2} \right) \right), \\ \tilde{\tilde{g}}_{zz} &= g_{rr} \left(\frac{2M}{r} - \frac{2Ma^2}{r^3} + \frac{a^4}{r^4} \right). \end{aligned} \quad (\text{B4})$$

Neglecting contributions of viscosity and heat in \mathbf{T} , the leading terms of $h_\alpha^z T_{;\beta}^{\alpha\beta} = 0$ are

$$G^{\alpha z} p_{;\alpha} + (\rho + p) u^\alpha u_{;\alpha}^z = 0. \quad (\text{B5})$$

The semicolon denotes covariant derivatives w.r.t. \mathbf{G} . This yields

$$G^{rz} p_{;r} + p_{;z} = -(\rho + p) \left[\Gamma_{tt}^z (u^t)^2 + \Gamma_{t\phi}^z u^t u^\phi + \Gamma_{\phi\phi}^z (u^\phi)^2 + \Gamma_{rr}^z (u^r)^2 \right], \quad (\text{B6})$$

where we already used $G^{zz} = 1 + \mathcal{O}(z/r)^2$ and $\Gamma_{tr}^z = \Gamma_{\phi r}^z = 0$. The remaining expressions in Eq. (B6) are

$$2\Gamma_{\alpha\beta}^z = -G_{\alpha\beta,z} - G^{rz} G_{\alpha\beta,r}, \quad \alpha, \beta \in \{t, \phi\}, \quad (\text{B7})$$

$$2\Gamma_{rr}^z = 2G_{rz,r} - G_{rr,z} + G^{rz} G_{rr,r}, \quad (\text{B8})$$

$$G^{rz} = -(z/r)\mathcal{Z} + \mathcal{O}(z/r)^2, \quad (\text{B9})$$

which leads to

$$\Gamma_{tt}^z = \frac{z}{r} \frac{M}{r^2} \mathcal{Y} + \mathcal{O}(z/r)^2, \quad (\text{B10})$$

$$\Gamma_{t\phi}^z = -\frac{z}{r} \frac{M}{r^2} a(2 + \mathcal{Y}) + \mathcal{O}(z/r)^2, \quad (\text{B11})$$

$$\Gamma_{\phi\phi}^z = \frac{z}{r} \frac{M}{r^2} [a^2(4 + \mathcal{Y}) + 2r^2] + \mathcal{O}(z/r)^2, \quad (\text{B12})$$

where \mathcal{Y}, \mathcal{Z} are given in Eq. (40). Equation (B6) is further reduced by $u^\phi = \Omega u^t$ with u^t as in Eq. (12) and the radial pressure gradient is, as in the case of the radial Euler equation (22) expressed in terms of c_{ad} ,

$$\frac{1}{\rho + p} \frac{dp}{dr} \simeq -c_{\text{ad}}^2 \left(\frac{u'}{u} + \frac{1}{r} \right). \quad (\text{B13})$$

This then leads to the final expression (38). The limiting value of \mathcal{R} at the horizon, $\tilde{\mathcal{R}}_{\text{h}}$,

$$\tilde{\mathcal{R}}_{\text{h}} = \frac{3}{4M^2} \left[(l_{\text{h}} - a u_{\text{h}} \mathcal{Z}_{\text{h}}^{1/2})^2 - l_{\text{h}}^2 (1 - \mathcal{Z}_{\text{h}}) \right] + \mathcal{Z}_{\text{h}}, \quad (\text{B14})$$

remains finite.

In Fig. 12 we plot three different expressions proposed for the relative half-thickness h/r , calculated for a typical type II solution using the exact expression for the shear (same solution as in Fig. 11). The solid line corresponds to Eq. (42), which is to be compared with the corresponding expressions derived from Eq. (43). These are given by $(h/r)^2 = (r/M)c_{\text{ad}}^2/\mathcal{R}$, with \mathcal{R} calculated using either γ instead of γ_k (short dashed line; L94) or γ_t instead of γ_k (long dashed line; ACGL96). All expressions yield similar flow morphology in the outer subsonic parts of the flow. The disc height (42) is finite at r_h , in agreement with (B14). Using γ in (43) overestimates h within $\sim 10r_g$, whereas γ_t yields an underestimate, with vanishing h at the horizon.

APPENDIX C: EXISTENCE AND LOCATION OF SONIC POINTS

In the Kerr metric circular equatorial geodesic orbits exist outside of the photon orbit $r_{\text{ph}}^\pm = 2M(1 + \cos(2/3 \cos^{-1}(\pm a/M)))$. At r_{ph} the Keplerian angular velocities Ω_k^\pm reach the limiting angular velocities Ω_\pm for timelike trajectories (dashed lines in Fig. 13, bottom). Circular geodesic orbits are unstable within the marginally stable orbit $r_{\text{ms}} > r_{\text{ph}}$. In the following we use abbreviations $\lambda_{\text{ph}} = \lambda_k(r_{\text{ph}})$ and $\lambda_{\text{ms}} = \lambda_k(r_{\text{ms}})$.

For given r_s, c_s the regularity condition $\mathcal{N}(\mathcal{D} = 0) = 0$ yields a quadratic in Ω_s ,

$$\begin{aligned} 0 &= a_0 + a_1\Omega_s + a_2\Omega_s^2, \\ a_0 &= c_s^2 r_s^2 (M - r_s)(4M^2 a^2 - r_s^2 \Delta) - MA\Delta, \\ a_1 &= 2MaA[\Delta - 2r_s c_s^2 (M - r_s)], \\ a_2 &= A[\Delta(r_s^3 - Ma^2) + Ac_s^2 (M - r_s)]. \end{aligned} \tag{C1}$$

In the limiting case $c_s = 0$, solutions of Eq. (C1) reduce to $\Omega_s^\pm = \Omega_k^\pm$, where $\Omega_s^+ \geq \omega$ corresponds to prograde and $\Omega_s^- < \omega$ to retrograde critical points. In the general case $0 < c_s^2 < 1/3$ one finds $\Omega_k^+ > \Omega_s^+$ for $r_s^+ > r_{\text{ph}}^+$ and $\Omega_k^- < \Omega_s^-$ for $r_s^- > r_{\text{ph}}^-$. Therefore transonic disc flows have sub-Keplerian Ω - and λ -distributions in the vicinity of the sonic point (Abramowicz & Zurek1981). A minimum critical radius $r_{s-}(c_s)$ exists, which is located between $r_{s-} = r_{\text{ph}}^\pm$ for hot flows with $c_s^2 = 1/3$ and $r_{s-} = r_{\text{ms}}^\pm$ for cold flows with $c_s = 0$. The corresponding λ represents an upper limit λ_{top} allowed for an ideal transonic polytropic flow with a given enthalpy (Kafatos & Yang1994). On the other hand, a maximum critical radius $r_{s+}(c_s)$ exists, which is smallest for hot flows and tends to infinity as c_s vanishes. In Fig. 13 we plot $\Omega_s^\pm(r_s)$ and $\lambda_s^\pm(r_s)$ for different values of c_s^2 as a parameter, both for a non-rotating and a rotating ($a = 0.95$) black hole. A limiting critical sound speed $c_s^*(a)$ separates the topology of the $\lambda_s(r_s)$ curves in two classes. If $c_s > c_s^*$, $\lambda_s(r_s)$ is monotonic, thus any $\lambda_s \leq \lambda_{\text{ph}}$ corresponds to one solution for r_s and $\lambda_s > \lambda_{\text{ph}}$ yields no solution at all. On the other hand, if $0 < c_s < c_s^*$, $\lambda_s(r_s)$ has a minimum λ_{min} and a maximum λ_{max} , which fall together for $c_s = c_s^*$. Obviously, $\lambda_{\text{min}} \leq \lambda_{\text{ms}}$ and $\lambda_{\text{max}} \geq \lambda_{\text{ms}}$. Consider first the case $\lambda_{\text{max}} \leq \lambda_{\text{ph}}$. Then for $\lambda_s \leq \lambda_{\text{min}}$ again exactly one r_s exists, for $\lambda_{\text{min}} < \lambda_s < \lambda_{\text{max}}$ there exist three solutions for r_s , and for $\lambda_s \geq \lambda_{\text{max}}$ again one r_s exists as long as $\lambda_s \leq \lambda_{\text{ph}}$, and no solution otherwise. The number of critical points in each of these three domains is reduced by one if $\lambda_{\text{max}} > \lambda_{\text{ph}}$. These critical points can be identified by the corresponding slope $d\lambda_s/dr_s$ and their relative location. An *outer* critical point $r_{s,\text{out}}$ exists under multiple critical point conditions and has $d\lambda_s/dr_s < 0$. An *intermediate* critical point $r_{s,\text{int}}$ has $d\lambda_s/dr_s > 0$. An *inner* critical point $r_{s,\text{in}}$ has $d\lambda_s/dr_s < 0$ and is located next to the horizon. The simple relation (C2) among r_s, c_s, Ω_s^\pm is locally relevant only. This changes for ideal flows if $\mathcal{N}(\mathcal{D} = 0) = 0$ is solved for conserved quantities, since then the solution topology is known from the boundary conditions. Out of numerous combinations we plot $\lambda_s^\pm(r_s)$ parametrized by E_s in Fig. 14 and $E_s(r_s)$ parametrized by λ_s in Fig. 15, both for $a = 0$ and for $a = 0.95$. Again, the same classes of critical points are recovered from both diagrams. For $E_s < E_s^*$ there exist extreme values λ_{min} and λ_{max} , such that for $\lambda_{\text{min}} < \lambda_s < \lambda_{\text{max}}$ multiple critical points occur. Formulated vice versa, for $\lambda_s \geq \lambda_s^*$ there exist extreme values E_{min} and E_{max} , such that for $E_{\text{min}} < E_s < E_{\text{max}}$ multiple critical points occur. This is summarized in Fig. 1, where $E_{\text{min}}(\lambda_s)$ and $E_{\text{max}}(\lambda_s)$ are plotted.

Consider now ideal flows with λ and E in the region of three critical points, and ask through which of these a particular transonic solution passes. The answer depends partly on the nature of critical points, which results from the critical slope $(du/dr)_s$, calculated by de l'Hôpital's rule from critical quantities. It turns out, that $r_{s,\text{int}}$ corresponds to an unphysical center-type critical point (spiral-type in viscous flows), whereas the *inner* and *outer* critical points are saddle-type, allowing for physical solutions. Thus, as λ increases continuously, the stationary solution 'jumps' from passing $r_{s,\text{out}}$ towards passing $r_{s,\text{in}}$ somewhere between λ_{min} and λ_{max} . Where this transition occurs in the E - λ -plane, depends on the modified accretion rate $\dot{\mathcal{M}}$ corresponding to a particular r_s . $\dot{\mathcal{M}}$ is generally different for solutions passing $r_{s,\text{in}}$ or $r_{s,\text{out}}$, $\dot{\mathcal{M}}(E, \lambda, r_{s,\text{in}}) \neq \dot{\mathcal{M}}(E, \lambda, r_{s,\text{out}})$. However, there exists an energy E_m , such that $\dot{\mathcal{M}}_m(E_m, \lambda, r_{s,\text{in}}) = \dot{\mathcal{M}}_m(E_m, \lambda, r_{s,\text{out}})$. The line BD in Fig. 1 represents the locus of these E_m .

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APPENDIX D: FIGURES

Figure 1: Division of the parameter space spanned by E_s , λ_s according to different types of ideal flow, for black holes with $a = 0$ (top diagram) and $a = 0.95$. Curve ABC is the boundary of the parameter space which separates the region (between AB and CD) with multiple critical points from the region with a single critical point (rest of diagram). BD is the locus of flows for which both, E_s and $\dot{\mathcal{M}}$, at the inner and at the outer critical points are the same. The dashed line indicates the maximum λ which an ideal flow with a given E may have in order to pass the centrifugal barrier. The locations of ideal models presented in section 5 are indicated by \times .

Figure 2: Sequence of ideal disc solutions around a non-rotating black hole. Solutions are for $a = 0$ and $E = 1.015$, with λ increasing (from bottom to top in the lower diagram) as $\lambda = 3.0, 3.25, 3.33, 3.335, 3.5, 3.8$. The sonic point is marked by \bullet . The location of these models in the E_s - λ_s -plane is marked by \times in Fig. 1.

Figure 3: Sequence of ideal disc solutions around a rotating black hole. Solutions are for $a = 0.95$ and $E = 1.06$, with λ increasing (from bottom to top in the lower diagram) as $\lambda = 2.0, 2.05, 2.08, 2.09, 2.15, 2.3$. The sonic point is marked by \bullet . The location of these models in the E_s - λ_s -plane is marked by \times in Fig. 1.

Figure 4: Sequence of type I ideal disc solutions around rotating black holes. Solutions are for $E = 1.2$ and $\lambda = 2.0$. The Kerr parameter a increases (from bottom to top) as $a = 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95$. The sonic point is marked by \bullet .

Figure 5: Sequence of type I viscous disc solutions around a non-rotating black hole. Solutions are for $a = 0$, $\dot{\mathcal{M}} = 0.032$ and $L_0 = 3.5$, with α decreasing (from top to bottom) as $\alpha = 0.045, 0.04, 0.03, 0.01$. Dashed curves represent Keplerian distributions. The sonic point is marked by \bullet .

Figure 6: Sequence of type I viscous disc solutions around a rotating black hole. Solutions are for $a = 0.95$, $\dot{\mathcal{M}} = 0.027$ and $\alpha = 0.022$, with L_0 decreasing (from top to bottom in the λ -diagram) as $L_0 = 2.43, 2.40, 2.30$. Dashed curves represent Keplerian distributions. The sonic point is marked by \bullet .

Figure 7: Sequence of type I viscous disc solutions around a non-rotating black hole. Solutions are for $a = 0$, $\dot{\mathcal{M}} = 0.032$ and $L_0 = 5.0$, with α decreasing (from top to bottom in the λ -diagram) as $\alpha = 0.008, 0.006$. Dashed curves represent Keplerian distributions. The sonic point is marked by \bullet .

Figure 8: Sequence of type I viscous disc solutions around a non-rotating black hole. Solutions are for $a = 0$, $\alpha = 0.001$ and $L_0 = 3.8$, with $\dot{\mathcal{M}}$ decreasing (from bottom to top in the λ -diagram) as $\dot{\mathcal{M}} = 0.0112, 0.0028, 0.0010, 0.0004$. Dashed curves represent Keplerian distributions. The sonic point is marked by \bullet .

Figure 9: Sequence of type II viscous disc solutions around a non-rotating black hole. Solutions are for $a = 0$ and $\dot{\mathcal{M}} = 0.011$. The upper two solutions have $\alpha = 0.6$ and (from top to bottom in the λ -diagram) $L_0 = 1.3, 1.1$. The lower two solutions have $\alpha = 0.5$ and (from top to bottom in the λ -diagram) $L_0 = 0.9, 0.0$. Dashed curves represent Keplerian distributions. The sonic point is marked by \bullet .

Figure 10: Sequence of type II viscous disc solutions around a rotating black hole. Solutions are for $a = 0.95$, $\dot{\mathcal{M}} = 0.011$ and $L_0 = 1.0$, with α decreasing (from top to bottom in the λ -diagram) as $\alpha = 0.8, 0.7, 0.3$. Dashed curves represent Keplerian distributions. The sonic point is marked by \bullet .

Figure 11: Different expressions for the shear tensor component σ_ϕ^r , plotted for the supersonic part of a typical type II solution ($a = 0$, $\alpha = 0.5$, $\dot{\mathcal{M}} = 0.017$, $L_0 = 0.9$). The solid line describes the full expression (A4), used to calculate the underlying disc solution. The short dashed line corresponds to approximation (A5). Other curves represent expressions obtained from Eq. (32) with γ and Ω' replaced by γ_k and Ω'_k (long dashed line) and with γ_t and Ω' replaced by γ_k and Ω'_k (short dash-long dashed line).

Figure 12: Different expressions for the relative half-thickness h/r , monitored for a typical type II solution ($a = 0$, $\alpha = 0.5$, $\dot{\mathcal{M}} = 0.017$, $L_0 = 0.9$). The solid line describes the full expression (42). Dashed lines correspond to h/r as derived from Eq. (43), with γ instead of γ_k (short dashed) and γ_t instead of γ_k (long dashed).

Figure 13: Parameter space showing contours of constant critical sound speed c_s . The critical angular velocity Ω_s^\pm (bottom) and the corresponding critical specific angular momentum λ_s^\pm (top) are plotted for non-rotating ($a = 0$; left diagrams) and for a rotating ($a = 0.95$) black hole. Contours correspond to (from left to right) $c_s^2 = 0.3, 0.15, 0.08, 0.04, 0.02, 0.0$. The $c_s^2 = 0.0$ curve is identical with the Keplerian distributions λ_k, Ω_k . In the Ω -diagrams, short-dashed lines represent the limiting angular frequencies Ω_\pm for timelike trajectories, and long-dashed lines represent the angular velocity ω of the LNRF.

Figure 14: Variation of critical specific angular momenta λ_s^\pm with the location of r_s for a set of parameters E_s . The upper diagram is for $a = 0$, the lower for $a = 0.95$. The outer dashed curves are the Keplerian distributions. Parameters E_s are (from the top curves to the bottom) $E_s = 0.98, 1.00, 1.02, 1.03, 1.05$ (dashed for $a = 0$), 1.10, 1.20, 1.50 respectively.

Figure 15: Variation of the critical energy E_s with the location of r_s for a set of parameters λ_s . The lower dashed curves are the Keplerian distributions. For $a = 0$ (upper diagram) parameters λ_s are (from the top curve to the bottom) $\lambda = 0, 2, 3, 3.13$ ($= \lambda^*$; dashed), 3.4, 3.6, 3.8, 4 and 4.5 respectively. For $a = 0.95$ (lower diagram) parameters λ_s are (from the top curve to the bottom) $\lambda_s = 0, 1.9, 2, 2.03$ ($= \lambda^*$; dashed), 2.1, 2.2, 2.3, 2.4 and 2.5 respectively. Further plotted (dashed lines) are the curves of extrema E_{\max} and E_{\min} .